Monotonic decrease of the quantum nonadditive divergence by projective measurements

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Nonadditive (nonextensive) generalization of the quantum Kullback-Leibler divergence, termed the quantum q-divergence, is shown not to increase by projective measurements in an elementary manner.

PACS: 03.65.Ud, 03.67.-a, 05.20.-y, 05.30.-d

In recent papers [1-3], we have developed a nonadditive generalization of information theory and have discussed its distinguished roles in the study of quantum entanglement extensively (see also, [4-8]). These works have primarily been concerned with the Tsallis nonadditive (nonextensive) entropy [9] and the associated generalized conditional entropy [1]. On the other hand, quite recently, the role of the generalized Kullback-Leibler divergence, termed the quantum *q*-divergence, has been examined as a measure of the degree of state purification [10]. There, an advantageous point of the quantum *q*-divergence over the ordinary quantum Kullback-Leibler divergence has been clarified.

In this article, we study the behavior of the quantum *q*-divergence under measurements, i.e., quantum operations. In particular, we present an elementary proof that the quantum *q*-divergence does not increase by projective measurements.

The quantum *q*-divergence is the relative entropy associated with the Tsallis entropy. The Tsallis entropy reads

$$S_q[\rho] = -\frac{1}{1-q}\mathrm{Tr}\left(\rho^q \ln_q \rho\right). \qquad (1)$$

Here, ρ is the normalized density matrix of the quantum system under consideration and q is the entropic index which can be an arbitrary positive number at this level. $\ln_q x$ stands for the *q*-logarithmic function [11] defined by $\ln_q x = (x^{1-q} - 1)/(1-q)$, which tends to the ordinary logarithmic function, $\ln x$, in the limit $q \to 1$. Then, the quantum q-divergence of ρ with respect to the reference density matrix σ is given by

$$K_q[\rho \,\|\, \sigma] = \mathrm{Tr}\left[\rho^q \left(\ln_q \rho - \ln_q \sigma\right)\right]. \tag{2}$$

(The classical counterpart of this quantity has been introduced independently and almost simultaneously in [12-14].) Using the definition of the q-logarithmic function, Eq. (2) can also be written in the following compact form:

$$K_q[\rho \,\|\, \sigma] = \frac{1}{1-q}\left[1 - \mathrm{Tr}\left(\rho^q \sigma^{1-q}\right)\right]. \tag{3}$$

Since this quantity should not be too sensitive to small eigenvalues of the density matrices, the range of q is taken to be

$$0 < q < 1. \tag{4}$$

Let s_ρ and s_σ be the supports of ρ and σ, respectively. In the case when $s_\rho \le s_\sigma$, $K_q[\rho \,\|\, \sigma]$ has the well-defined limit $q \to 1-0$, which yields the ordinary quantum Kullback-Leibler divergence introduced by Umegaki [15]

$$K[\rho \,\|\, \sigma] = \mathrm{Tr}\left[\rho\left(\ln\rho - \ln\sigma\right)\right]. \tag{5}$$

Here, the condition, $s_\rho \le s_\sigma$, is crucial. In fact, $K[\rho \,\|\, \sigma]$ becomes singular when $s_\rho > s_\sigma$. Therefore, $K[\rho \,\|\, \sigma]$ cannot be defined if σ is a pure state (i.e., an idempotent operator), for example. In marked contrast to this, $K_q[\rho \,\|\, \sigma]$ with $q \in (0,1)$ remains well-defined even in such a case [10].

In Ref. [10], it has been shown that (i) $K_q[\rho \,\|\, \sigma] \ge 0$ and $K_q[\rho \,\|\, \sigma] = 0$ if and only if $\rho = \sigma$, (ii) for product states, $\rho(A, B) = \rho_1(A) \otimes \rho_2(B)$ and $\sigma(A, B) = \sigma_1(A) \otimes \sigma_2(B)$, of a bipartite system (A, B), $K_q[\rho \,\|\, \sigma]$ satisfies pseudoadditivity: $K_q[\rho_1 \otimes \rho_2 \,\|\, \sigma_1 \otimes \sigma_2] = K_q[\rho_1 \,\|\, \sigma_1] + K_q[\rho_2 \,\|\, \sigma_2] + (q-1) K_q[\rho_1 \,\|\, \sigma_1] K_q[\rho_2 \,\|\, \sigma_2]$ and (iii) K_q can be observed as the q-analog (i.e., q-deformation) of K in the sense in [16].

In addition to the properties (i)-(iii), we wish to notice another important one anew here. That is, $K_q[\rho \,\|\, \sigma]$ is jointly convex

$$K_q\left[\sum_i \lambda_i\, \rho^{(i)} \,\|\, \sum_i \lambda_i\, \sigma^{(i)}\right] \le \sum_i \lambda_i\, K_q\left[\rho^{(i)} \,\|\, \sigma^{(i)}\right], \tag{6}$$

where $\lambda_i > 0$ and $\sum_i \lambda_i = 1$. This directly follows from the expression in Eq. (3) as well as Lieb's theorem [17] stating that $\mathrm{Tr}\left(L^{1-x} M^{x}\right)$ with $x \in (0,1)$ is jointly concave in any positive operators, L and M. Eq. (6) generalizes joint convexity of the ordinary quantum divergence (see [18], for example).

Now, let us discuss the behavior of $K_q[\rho \| \sigma]$ under projective measurement of ρ and σ. This measurement can be regarded as a particular kind of positive trace-preserving quantum operation, but is quite common from the experimental viewpoint [19]. Let Q be an observable with eigenspaces defined by orthogonal projections P_k and $\{q_k\}$ be its measured values. Then, $Q = \sum_k q_k P_k$, $P_k P_{k'} = \delta_{kk'} P_{k'}$ and $\sum_k P_k = I$. The finite probability p_k of obtaining the value q_k of Q in a state ρ of the system through the projective measurement is $p_k = \mathrm{Tr}\left(\rho P_k\right)$. From this, ρ is transformed to $\rho_k = p_k^{-1} P_k \rho P_k$. Averaging over all possible outcomes, we have

$$\Pi(\rho) = \sum_k p_k \rho_k = \sum_k P_k \rho P_k . \tag{7}$$

Clearly, Π is a positive trace-preserving operation.

Let us employ the diagonal representations of ρ and σ:

$$\rho = \sum_a r(a) |a\rangle\langle a|, \qquad \sigma = \sum_b s(b) |b\rangle\langle b|, \tag{8}$$

where $r(a) \geq 0$, $\sum_a r(a) = 1$, $\langle a | a' \rangle = \delta_{aa'}$, $\sum_a |a\rangle\langle a| = I$ and so on. Under the operation of a projective measurement, they are replaced by

$$\Pi(\rho) = \sum_a r(a)\, \Pi\big(|a\rangle\langle a|\big), \qquad \Pi(\sigma) = \sum_b s(b)\, \Pi\big(|b\rangle\langle b|\big), \tag{9}$$

respectively. Let us further use the diagonal representations

$$\Pi\big(|a\rangle\langle a|\big) = \sum_\alpha \mu(\alpha, a)\, |\alpha\rangle\langle\alpha|, \qquad \Pi\big(|b\rangle\langle b|\big) = \sum_\beta \nu(\beta, b)\, |\beta\rangle\langle\beta|, \tag{10}$$

where $\mu(\alpha, a) = \sum_k \left|\langle\alpha|\, P_k\, |a\rangle\right|^2 \geq 0$, $\sum_a \mu(\alpha, a) = \sum_\alpha \mu(\alpha, a) = 1$, $\langle\alpha|\alpha'\rangle = \delta_{\alpha\alpha'}$, $\sum_\alpha |\alpha\rangle\langle\alpha| = I$ and so on. Accordingly, we have

$$[\Pi(\rho)]^q = \sum_{a,\alpha} [r(a)\, \mu(\alpha, a)]^q\, |\alpha\rangle\langle\alpha|, \; [\Pi(\sigma)]^{1-q} = \sum_{b,\beta} [s(b)\, \nu(\beta, b)]^{1-q}\, |\beta\rangle\langle\beta|, \tag{11}$$

which leads to

$$\mathrm{Tr}\left\{[\Pi(\rho)]^q\, [\Pi(\sigma)]^{1-q}\right\} = \sum_{a,\alpha} \sum_{b,\beta} [r(a)\, \mu(\alpha, a)]^q\, [s(b)\, \nu(\beta, b)]^{1-q} \left|\langle\alpha|\beta\rangle\right|^2. \tag{12}$$

Since $0 \leq \mu, \nu \leq 1$ and $0 < q < 1$, we see that $\mu^q > \mu$, $\nu^q > \nu$. Therefore, we have

$$\mathrm{Tr}\left\{[\Pi(\rho)]^q\, [\Pi(\sigma)]^{1-q}\right\} \geq \sum_{a,\alpha} \sum_{b,\beta} [r(a)]^q\, [s(b)\,]^{1-q}\, \mathrm{Tr}\big[\Pi\big(|a\rangle\langle a|\big)\, \Pi\big(|b\rangle\langle b|\big)\big]. \tag{13}$$

Noting that

$$\mathrm{Tr}\left[\Pi\left(|a\rangle\langle a|\right)\Pi\left(|b\rangle\langle b|\right)\right] = \sum_{k,\,k'} \left|\langle a|P_k\ P_{k'}\,|b\rangle\right|^2$$

$$\geq \left|\sum_{k,\,k'} \langle a|P_k\ P_{k'}\,|b\rangle\right|^2 = \left|\langle a|b\rangle\right|^2, \tag{14}$$

we find

$$\mathrm{Tr}\left\{\left[\Pi(\rho)\right]^q \left[\Pi(\sigma)\right]^{1-q}\right\} \geq \mathrm{Tr}\left(\rho^q\ \sigma^{1-q}\right), \tag{15}$$

leading to

$$K_q\left[\Pi(\rho)\|\Pi(\sigma)\right] \leq K_q\left[\rho\|\sigma\right]. \tag{16}$$

Therefore, we obtain the main result that the quantum q-divergence does not increase by projective measurements.

In conclusion, we have shown that the quantum q-divergence is jointly convex and does not increase by projective measurements.

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